A probabilistic power domain of evaluations – Notes

Based on [AJ94, JP89], presented by Dominique Unruh on 2008-01-15.

1 Order theory

Partial order. A partial order $\leq$ on a set X is a relation that is reflexive ($x \leq x$), transitive ($x \leq y \wedge y \leq z \Rightarrow x \leq z$), and antisymmetric ($x \leq y \wedge y \leq x \Rightarrow x = y$).

Natural ordering (on functions). For a partially ordered set X, the natural ordering of $M \to X$ (the set of functions $M \to X$) is given by $f \leq g \iff \forall x. f(x) \leq g(x)$.

Directed set. A partially ordered set M is *directed* if for any $x, y \in M$ there exists an upper bound in M (i.e., an $z \in M$ with $z \geq x$ and $z \geq y$).

Least upper bound (lub). A *least upper bound* of M (in N) is an element $b \in N$ such b is an upper bound for M (i.e., $b \geq x$ for all $x \in M$) and such that for any upper bound $b' \in N$ for M, we have $b \leq b'$. We write $\bigvee M$ for the lub of M.

Directed-complete partial order (dcpo). A partially ordered set D is a *dcpo* if any directed subset $M \subseteq D$ has a lub in D.

Scott-continuous. A function $f : D \to E$ between two dcpos is called *continuous* if it is monotonous and preserves lubs of directed sets (i.e., for any directed $M \subseteq D$ we have that $f(\bigvee M) = \bigvee f(M)$).

Scott-topology. A subset M of a dcpo D is called *Scott-closed* if it is downward closed (i.e., for $x \in M$ and $y \leq x$ we have $y \in M$) and closed under lubs of directed subsets (i.e., for any directed $N \subseteq M$ we have $\bigvee N \in M$).

A subset M is called *Scott-open* if its complement is Scott-closed.

A function f is *Scott-continuous* iff it is continuous with respect to the Scott-topology (i.e., if for any open $O \subseteq E$, we have that $f^{-1}(O)$ is open in D).

2 Category theory

Category. A *category* $\mathbf{C}$ consists of *objects* together with a class $\mathrm{Hom}(D, E)$ of *morphisms* for each pair D, E of objects. We write $f : D \to E$ for $f \in \mathrm{Hom}(D, E)$. We assume that there is a associative operation $\circ$ (*composition*) such that for $f : D \to E$ and $g : E \to F$ we have $g \circ f : D \to E$. Further for every object D we assume a morphism $id_D : D \to D$ such that for all morphisms f, g we have $f \circ id_D = f$ and $id_D \circ g = g$.

Often (but not necessarily), the objects are sets enriched with some structure (e.g., dcpos), and the morphisms are functions between these sets that satisfy some properties (e.g., Scott-continuity).

Functor. A *functor* F between categories $\mathbf{C}$ and $\mathbf{D}$ assigns to each object A of $\mathbf{C}$ an object FA of $\mathbf{D}$. Further, to each morphism $f : A \to B$ of $\mathbf{C}$ it assigns a morphism $Ff : FA \to FB$. We require that $Fid = id$ and $F(f \circ g) = Ff \circ Fg$.

Example: The *powerset functor* on the category of sets assigns to a set M its powerset 2^M, and maps $f : M \to N$ to the function $Ff : 2^M \to 2^N$ with $Ff(U) = \{f(x) : x \in U\}$ for all $U \in 2^M$.

3 Recursive Domain Equations

Locally continuous functor. Let a functor F between categories $\mathbf{D}$ and $\mathbf{E}$ of dcpos be given. For objects X, Y of $\mathbf{D}$, defined the function $F_{X,Y} : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(FX, FY)$ by $F_{X,Y}(f) = Ff$. Assume the natural ordering on $\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(FX,FY)$.

We call F *locally continuous* if for all X, Y, $F_{X,Y}$ an isomorphism (i.e., it is continuous and has a continuous inverse).

Continuous functor. We omit this definition. However, every locally continuous functor is continuous.

Theorem 1: Solving recursive domain equations. Let F be a locally continuous functor from the category of dcpos to the category of dcpos. Then there is a dcpo X with $X \cong FX$. (Note that there are many additional properties this solution fulfils, e.g., it is the initial solution, etc. We omit the properties here.)

4 Probabilistic Power domain

Continuous evaluation. Let X be a topological space and $\mathcal{O}(X)$ be the set of all open subsets of X. A function $\mu : \mathcal{O}(X) \to [0,1]$ is called a *continuous evaluation* if:

- *Monotonicity:* If $U \subseteq V$ then $\mu(U) \leq \mu(V)$.
- *Strictness:* $\mu(\varnothing) = 0$.
- *Modularity:* $\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V)$.
- *Continuity:* For a directed collection of open sets $U_\lambda \in \mathcal{O}(X)$, we have that $\mu(\bigcup_\lambda U_\lambda) = \bigvee_\lambda (\mu(U_\lambda))$.

"Probabilistic identity" $\boldsymbol{\eta}$. For $x \in M$, let $\eta(x)(M) = 1$. For $x \notin M$, let $\eta(x)(M) = 0$.

Probabilistic power domain. For any dcpo X, let $\mathcal{E}X$ denote the dcpo of all continuous evaluations on X. For a function $f : X \to Y$, let $\mathcal{E}f : \mathcal{E}X \to \mathcal{E}Y$ be defined by $\mathcal{E}f(\mu)(M) = \mu(f^{-1}(X))$ (that is, $\mathcal{E}f(\mu)$ is the "distribution" resulting from sampling according to μ and applying f to the outcome).

We call $\mathcal{E}X$ the *probabilistic power domain.*

Theorem 2: Solving probabilistic recursive domain equations. $\mathcal{E}$ is a locally continuous functor on the category of dcpos. Thus Theorem 1 can be applied.

References

[AJ94] Samson Abramsky and Achim Jung. Domain theory. In S. Abramsky, D. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science Volume 3*, pages 1–168. Oxford University Press, 1994. Extended version available online at `http://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf`.

[JP89] C. Jones and Gordon D. Plotkin. A probabilistic powerdomain of evaluations. In *LICS*, pages 186–195. IEEE Computer Society, 1989. Available online at `http://dx.doi.org/10.1109/LICS.1989.39173`.